LMU-10/95
April, 1995

Comments on the Characteristic Measurables of the Quark Mixing Matrix

Zhi-zhong XING [1]

Sektion Physik, Theoretische Physik, Universität München,
Theresienstrasse 37, D-80333 München, Germany

Abstract

Within the standard electroweak model we point out that the 3×3 matrix of quark mixing is characterized by three universal (rephasing-invariant) quantities: one of them for CP violation and the other two for off-diagonal asymmetries. Unitarity of the quark mixing matrix can in principle be tested through a variety of measurements which are irrelevant to the existence of CP-violating effects.

PACS numbers: 12.15.Ff, 11.30.Er, 13.25.Hw

[1] E-mail address: Xing@hep.physik.uni-muenchen.de

In the standard electroweak model, the 3×3 Cabibbo-Kobayashi-Maskawa (CKM) matrix V provides a natural description of quark mixing and CP violation [1,2]. Unitarity is the only but powerful constraint, imposed by the model itself, on V. This restriction is commonly expressed as two sets of orthogonality-plus-normalization conditions:

$$\sum_{k=1}^{3} V_{ik} V^*_{jk} \;=\; \delta_{ij} \; , \qquad\qquad \sum_{i=1}^{3} V_{ij} V^*_{ik} \;=\; \delta_{jk} \; , \tag{1}$$

where $i, j, k = 1, 2, 3$, running over the up-type quarks (u, c and t) and the down-type quarks (d, s and b). In the complex plane the six orthogonality relations given above correspond to six triangles (see Fig. 1), the so-called unitarity triangles [3]. Confronting these unitarity requirements with the existing and forthcoming experimental data may serve for a stringent test of the standard model.

By use of the unitarity conditions in Eq. (1), one can parametrize the CKM matrix in various ways. Several popular parametrizations [2,4,5], including the "standard" one [6], are given in terms of three Euler angles and one CP-violating phase. It is interesting to note the fact that knowledge of only the magnitudes of four independent V_{ij} is sufficient to determine all phase information and construct the entire matrix V [7]. On the other hand, four independent angles (inner or outer) of the six unitarity triangles, once they are measured from the CP asymmetries in weak B-meson decays, can also determine the whole quark mixing matrix [8,9].

In this note we shall point out that the CKM matrix V is in fact characterized by three universal (rephasing-invariant) quantities: one of them for CP violation and the other two for the off-diagonal asymmetries of V. These measurables have interesting relations with the six unitarity triangles. We briefly comment on some possibilities to test unitarity of the quark mixing matrix through the accessible measurements at present or in the near future. We stress the point that the Kobayashi-Maskawa mechanism of CP violation [2] can experimentally be checked even in the absence of direct observation of CP-violating signals.

It is well known that unitarity of the CKM matrix leads to a universal and rephasing-invariant measure of CP violation for quark weak interactions [10,11], the so-called Jarlskog parameter J [10]:

$$\mathrm{Im}\left(V_{il} V_{jm} V^*_{im} V^*_{jl}\right) \;=\; J \sum_{k,n=1}^{3} \epsilon_{ijk} \epsilon_{lmn} \; . \tag{2}$$

One can show that all the six unitarity triangles have the same area $J/2$, although their shapes are quite different (see Fig. 1 for illustration). Here the interesting point is that J^2 can be simply expressed in terms of three sides of each triangle:

$$J^2 \;=\; 4P_i \prod_{l=1}^{3} \left(P_i - |V_{jl} V^*_{kl}|\right) \;=\; 4Q_i \prod_{l=1}^{3} \left(Q_i - |V_{lj} V^*_{lk}|\right) \; , \tag{3a}$$

where the subscripts i, j and k (=1,2,3) must be co-cyclic for the up- or down-type quarks, P_i and Q_i are given by

$$P_i \; = \; \frac{1}{2} \sum_{l=1}^{3} |V_{jl} V^*_{kl}| \; , \qquad Q_i \; = \; \frac{1}{2} \sum_{l=1}^{3} |V_{lj} V^*_{lk}| \; . \tag{3b}$$

Clearly P_i and Q_i correspond to the unitarity triangles $[u], [c], [t]$ and $[d], [s], [b]$ in Fig. 1, respectively. This result implies that one can in principle obtain the information about CP violation only from the sides of the unitarity triangles. In this way we are able to experimentally check unitarity of the quark mixing matrix and the Kobayashi-Maskawa picture of CP violation, without the help of direct measurements of any CP-violating signal.

The off-diagonal asymmetries of the CKM matrix are another two universal and characteristic quantities for quark mixing. They are denoted by Z_1 about the $V_{11} - V_{22} - V_{33}$ axis [12] and Z_2 about the $V_{13} - V_{22} - V_{31}$ axis:

$$|V_{ij}|^2 - |V_{ji}|^2 \; = \; Z_1 \sum_{k=1}^{3} \epsilon_{ijk} \; , \qquad |\hat{V}_{ij}|^2 - |\hat{V}_{ji}|^2 \; = \; Z_2 \sum_{k=1}^{3} \epsilon_{ijk} \; , \tag{4a}$$

where the matrix $\hat{V}$ is obtained from V through the following rotation:

$$\hat{V} \; = \; VR \; , \qquad R \; = \; \left(\begin{matrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{matrix} \right) \; . \tag{4b}$$

Certainly the axis $\hat{V}_{11} - \hat{V}_{22} - \hat{V}_{33}$ of $\hat{V}$ is equivalent to the axis $V_{13} - V_{22} - V_{31}$ of V. The above result can be shown easily by use of the normalization conditions of unitarity given in Eq. (1). Note that the asymmetry parameters $Z_{1,2}$ are independent of each other, and they are independent of the CP-violating parameter J. Although a little attention was paid to Z_1 in the literature [12,13], Z_2 has been ignored. We shall see later on that $Z_2 >> Z_1$, and both of them play interesting roles in testing unitarity of the quark mixing matrix.

Explicitly One can express J and $Z_{1,2}$ with any set of parameters suggested in Refs. [2,4-6]. To illustrate these characteristic measurables in a simple and instructive way, here we use the Wolfenstein parameters [4]. Note that a self-consistent calculation of $Z_{1,2}$ can only be carried out on the basis of the modified Wolfenstein parametrization [12] [2], in which unitarity is kept up to the accuracy of $O(\lambda^6)$. In terms of λ, A, ρ and η [12,14], we obtain

$$Z_1 \; \approx \; A^2 \lambda^6 (1 - 2\rho) \; , \qquad Z_2 \; \approx \; \lambda^2 \left(1 - A^2 \lambda^2 \right) \; , \qquad J \; \approx \; A^2 \lambda^6 \eta \; . \tag{5}$$

[2] In terms of the Wolfenstein parameters, Kobayashi has presented a parametrization of the CKM matrix with exact unitarity [14]. An incomplete modification of the Wolfenstein parametrization, where the imaginary parts of V_{21} and V_{32} are corrected up to the accuracy of $O(\lambda^5)$, was given by Buras *et al* for their own purpose in Ref. [15]. About seven years before, Branco and Lavoura have parametrized the CKM matrix up to $O(\lambda^8)$ by taking $V_{12} = \lambda, V_{23} = A\lambda^2$ and $V_{13} = A\mu\lambda^3 e^{i\phi}$ [16].

It is clear that $Z_2 >> Z_1$ and $Z_1 \sim J$. Both Z_1 and Z_2 are independent of η, a parameter necessary for signalling CP violation. Considering the values of λ, A, ρ and η extracted from experiments [17], we find $Z_2/Z_1 \geq 400$, $Z_1 \sim 10^{-5} - 10^{-4}$ and $J \sim 10^{-5}$. The possibility of $Z_1 \approx 0$, which requires $\rho \approx 0.5$, is only allowed on the margin of the current data [12].

We are in a position to give a few comments on the measure of CP violation, the unitarity triangles and the possibilities to test unitarity of the CKM matrix:

(a) Although all measurables of CP violation are proportional to J in the standard model, their magnitudes are more sensitively related to the angles of the unitarity triangles (see, e.g., Refs. [8,9]). This is why different weak processes of quarks may have different sizes of CP-violating effects. In this sense whether the value of J is maximal or not has less physical significance than the maximal violation of P (parity) symmetry does.

(b) In general the six unitarity triangles have nine different inner (or outer) angles, although they have eighteen different sides (see Fig. 1). If $Z_1 = 0$ holds, one can find three equivalence relations among the six triangles:

$$[u] \cong [d] , \qquad [c] \cong [s] , \qquad [t] \cong [b] . \tag{6}$$

In this case the six unitarity triangles have six different inner (or outer) angles and nine different sides. As a consequence the CKM matrix can be parametrized by use of three independent quantities. If we further assume $Z_2 = Z_1 = 0$, then two sides of the triangle $[c]$ or $[s]$ would become equal (i.e., $|V_{31}V_{11}^*| = |V_{33}V_{13}^*|$ etc). Today the possibility of $Z_2 = 0$ has been ruled out absolutely, while that of $Z_1 = 0$ is still in marginal agreement with the experimental restriction [3].

(c) If $Z_1 > 0$ is really true, then one can find the following hierarchical relation among the nine matrix elements:

$$|V_{33}| > |V_{11}| > |V_{22}| >> |V_{12}| > |V_{21}| >> |V_{23}| > |V_{32}| >> |V_{31}| > |V_{13}| . \tag{7}$$

This interesting result reflects the unitarity of the CKM matrix in an indirect way.

(d) It is instructive to express the nine inner angles of the unitarity triangles in terms of the Wolfenstein parameters. To lowest order approximation, we obtain the following results (see Fig. 1):

$$\tan(\angle 1) \approx \tan(\angle 4) \approx -\tan(\angle 3) \approx \frac{\eta}{1-\rho} , \tag{8a}$$

$$\tan(\angle 6) \approx \tan(\angle 7) \approx -\tan(\angle 9) \approx \frac{\eta}{\rho} , \tag{8b}$$

[3] The current data have given $|V_{12}| = 0.2205 \pm 0.0018$ and $|V_{21}| = 0.204 \pm 0.017$ [6], which implies a dominant possibility of $Z_1 > 0$.

and

$$\tan(\angle 2) \approx \lambda^2 \eta \, , \qquad \tan(\angle 8) \approx A^2 \lambda^4 \eta \, , \qquad \tan(\angle 5) \approx \frac{\eta}{\rho(\rho - 1) + \eta^2} \, . \tag{8c}$$

Conventionally one uses $\alpha = \angle 5$, $\beta = \angle 1$ and $\gamma = \angle 7$ to denote the three angles of unitarity triangle [s], which will be overdetermined at B-meson factories. We expect that the approximate relations given in Eq. (7) can be tested in various experiments of CP violation and B physics in the near future [18].

(e) The magnitude of J is now determinable from the unitarity triangle [t], whose three sides have all been measured in weak decays of the relevant quarks (or from deep inelastic neutrino scattering [6]). The off-diagonal asymmetries of V can be directly determined from the experimental data on V_{12}, V_{21} and V_{23}, i.e., $Z_1 = |V_{12}|^2 - |V_{21}|^2$ and $Z_2 = |V_{12}|^2 - |V_{23}|^2$. With the help of precise information about $B^0_d - \bar{B}^0_d$ mixing, one is able to determine $|V_{33}V^*_{31}|$ and then to establish the unitarity triangle [s]. A comparison between the areas of [t] and [s] may serve to confirm the unitarity conditions of the CKM matrix. Similarly the forthcoming measurement of $B^0_s - \bar{B}^0_s$ mixing will determine $|V_{32}V^*_{33}|$ and construct the triangle [d]. It might be a long run to directly measure $|V_{31}|$ and $|V_{32}|$ from the production or decay processes of the top quark.

(f) The normalization relations of unitarity (see Eq. (1)) can be well checked after more precise determination of $|V_{13}|$ from charmless B-meson decays and measurements of $|V_{33}|$ from the top-quark lifetime. To test unitarity of the CKM matrix up to $O(\lambda^6)$, of course, much effort is needed to make in order to improve the accuracy of the six elements in the first two rows of V. In practice observation of CP violation in B-meson decays will provide a good chance to judge the Kobayashi-Maskawa mechanism of CP violation as well as the six orthogonality conditions of unitarity. Discussions about violation of unitarity of the 3×3 CKM matrix, e.g., in the presence of the fourth-family quarks or an exotic charge $-1/3$ quark, would be beyond the scope of this note [19].

In summary, we have pointed out that the 3×3 matrix of quark mixing is characterized by three universal observables: the measure of CP violation J and the off-diagonal asymmetries $Z_{1,2}$. These three parameters have interesting relations with the six unitarity triangles. The Kobayashi-Maskawa picture of CP violation can in principle be examined through a variety of measurements irrelevant to the existence of CP-violating effects. A complete test of unitarity of the CKM matrix (to a good degree of accuracy) is accessible in the near future.

The author would like to thank Professor H. Fritzsch for his warm hospitality and constant encouragement. He is also grateful to Professors G. C. Branco and D. D. Wu for useful discussions, and to Professor R. E. Shrock for helpful comments. This work was financially supported by the Alexander von Humboldt Foundation of Germany.

References

[1] N. Cabibbo, Phys. Rev. Lett. 10, 531 (1963).

[2] M. Kobayashi and T. Maskawa, Prog. Theor. Phys. 49, 552 (1973).

[3] J. L. Rosner, A. I. Sanda, and M. P. Schmidt, in *Proceedings of the Workshop on High Sensitivity Beauty Physics at Fermilab*, Fermilab, November 1987, edited by A. J. Slaughter, N. Lockyer, and M. Schmidt (Fermilab, Batavia, IL, 1988), p. 165;
C. Hamzaoui, J. L. Rosner, and A. I. Sanda, in the same proceedings as above, p. 215;
Y. Nir and H. R. Quinn, Annu. Rev. Nucl. Part. Sci. 42, 211 (1992);
A. Ali, Report No. CERN-TH.7123/93 (talk given in the Conference on Highlights of Particle and Condensed Matter Physics — *Salamfest*, Trieste, March 1993).

[4] L. Wolfenstein, Phys. Rev. Lett. 51, 1945 (1983).

[5] H. Fritzsch and J. Plankl, Phys. Rev. D35, 1732 (1987);
H. Harari and M. Leurer, Phys. Lett. B181, 123 (1986);
L. L. Chau and W. Y. Keung, Phys. Rev. Lett. 53, 1802 (1984);
L. Maiani, Phys. Lett. B62, 183 (1976).

[6] F. J. Gilman, K. Kleinknecht, and B. Renk, in *Review of Particle Properties*, Phys. Rev. D50, 1315 (1994).

[7] J. D. Bjorken, Phys. Rev. D39, 1396 (1989);
C. Jarlskog and R. Stora, Phys. Lett. B208, 268 (1988);
G. C. Branco and L. Lavoura, Phys. Lett. B208, 123 (1988);
C. Hamzaoui, Phys. Rev. Lett. 61, 35 (1988).

[8] R. Aleksan, B. Kayser, and D. London, Phys. Rev. Lett. 73, 18 (1994).

[9] D. D. Wu and Z. Z. Xing, Phys. Lett. B341, 386 (1995).

[10] C. Jarlskog, Phys. Rev. Lett. 55, 1039 (1985).

[11] D. D. Wu, Phys. Rev. D33, 860 (1986);
J. Botella and L. L. Chau, Phys. Lett. B168, 97 (1986);
I. Dunietz, O. W. Greenberg, and D. D. Wu, Phys. Rev. Lett. 55, 2935 (1985);
O. W. Greenberg, Phys. Rev. D32, 1841 (1985);
A general scheme to construct new invariants of quark mass matrices and to determine their physically meaningful phases is given by A. Kusenko and R. E. Shrock in, e.g., Phys. Rev. D50, R30 (1994).

[12] Z. Z. Xing, Phys. Rev. D51, 3958 (1995).

[13] G. C. Branco and P. A. Parada, Phys. Rev. D44, 923 (1991);
G. Bélanger, C. Hamzaoui, and Y. Koide, Phys. Rev. D45, 4186 (1992).

[14] M. Kobayashi, Prog. Theor. Phys. 92, 287 (1994); 92, 289 (1994).

[15] A. J. Buras, M. E. Lautenbacher, and G. Ostermaier, Phys. Rev. D50, 3433 (1994).

[16] G. C. Branco and L. Lavoura, Phys. Rev. D38, 2295 (1988).

[17] A. Ali and D. London, Z. Phys. C65, 431 (1995).

[18] S. Stone, Nucl. Inst. and Meth. A333, 15 (1993);
Y. Sakai, Nucl. Inst. and Meth. A333, 42 (1993);
T. Nakada, Report No. CERN 90-2 (1990);
BaBar Collaboration, Report No. SLAC-443 (1994); SLAC-400 (1992).

[19] For a brief review of deviation behaviour of the unitarity triangles in some nonstandard electroweak models, see, e.g., Y. Nir and H. R. Quinn, in Ref. [3].

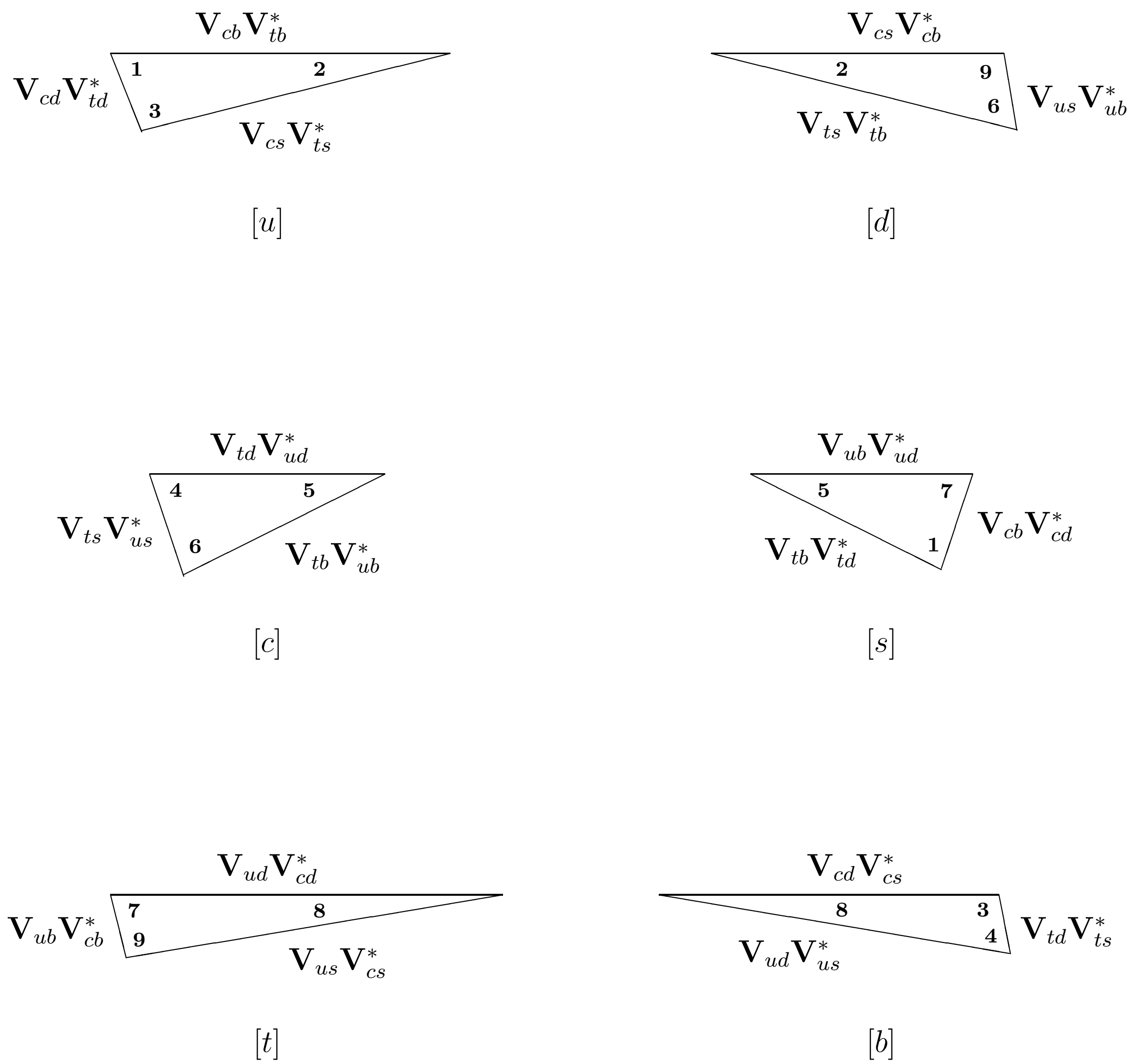

Figure 1: The unitarity triangles of the CKM matrix in the complex plane. Each triangle is named in terms of the quark flavor that does not manifest in its three sides. Note that the six triangles have the same area, and they only have nine different inner angles (versus eighteen different sides).